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Generalized parabolic functions on white noise space

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Abstract

We study the positive solutions of a heat equation on an infinite-dimensional state space using Hida's white noise analysis. We establish an integral representation theorem for generalized parabolic functions via so-called generalized Cameron–Martin densities, and we apply the representation formula in the study of the positive generalized parabolic functions on the white noise space.

Keywords: Heat equation; Hida functional; Parabolic function; Positive functional; White noise space

1. Introduction and preliminaries

The study of harmonic functions and parabolic functions (i.e. solutions to a heat equation) on a finite-dimensional, smooth manifold is one of the main aspects of many research fields: harmonic analysis, potential theory, partial differential equations, analysis on manifolds and, etc. However there are few literatures on parabolic functions on an infinite-dimensional state space. Recently, Röckner (1992) has studied the Martin boundary of the Ornstein–Uhlenbeck semigroup on an abstract Wiener space, and established an integral representation theorem for positive, square integrable parabolic functions.

In this paper we give a new approach to positive generalized parabolic functions of the Ornstein–Uhlenbeck operator in the framework of Hida's white noise analysis. Our method in this paper is quite different from those in Röckner (1992). The motivation to introduce generalized parabolic functions is due to the fact that the structure of parabolic functions on an infinite-dimensional state space is very different from the case of finite-dimensional manifolds. Indeed, it was shown in Röckner (1992) that not every positive parabolic function of the Ornstein–Uhlenbeck semigroup can be represented via the Cameron–Martin densities. Therefore, it is needed to find out those points on the parabolic Martin boundary which are not given by the Cameron–Martin densities.

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In Section 2, we establish an integral representation theorem for a class of positive generalized parabolic functions via generalized Cameron–Martin densities. In Section 3, several integral representation formulas for proper nonnegative parabolic functions are derived. In Section 4, we prove that a positive generalized parabolic function is a solution to the heat equation, and we end this paper by showing that the generalized Cameron–Martin densities are the extreme points.

In the rest of this section, we recall several notions and basic results in the white noise analysis, and establish notations as well; for details, cf. Hida et al. (1993), Kuo (1992) and Yan (1993).

Let $\mathscr{S}(\mathbb{R}^d)$ be the Schwartz space of real-valued rapidly decreasing functions on $\mathbb{R}^d$, and let $\mathscr{S}'(\mathbb{R}^d)$ be its dual space. Denote by $(\mathscr{S}'(\mathbb{R}), \mathscr{B}(\mathscr{S}'(\mathbb{R})), \mu)$ the white noise space, that is, $\mathscr{B}(\mathscr{S}'(\mathbb{R}))$ is the Borel σ-field on $\mathscr{S}'(\mathbb{R})$ and μ is the white noise measure: the unique probability on $(\mathscr{S}'(\mathbb{R}), \mathscr{B}(\mathscr{S}'(\mathbb{R})))$ such that

$$\int_{\mathscr{S}'(\mathbb{R})} e^{i\langle x,\xi\rangle} d\mu(x) = e^{-(1/2)|\xi|_2^2} \quad \forall \xi \in \mathscr{S}(\mathbb{R}),$$

where $|\xi|_2$ denotes the $L^2(\mathbb{R})$ norm of ξ.

Let A be the self-adjoint operator on $L^2(\mathbb{R})$:

$$Af(x) = -\frac{d^2 f(x)}{dx^2} + (1+x^2)f(x) \quad \forall f \in \mathscr{S}(\mathbb{R}),$$

and for each $p \in \mathbb{R}$, let

$$\mathscr{S}_p(\mathbb{R}^d) \triangleq \{f \in \mathscr{S}'(\mathbb{R}^d)\colon |f|_{2,p}^2 < \infty\},$$

where $|f|_{2,p} = |(A^{\otimes d})^p f|_2$. For simplicity, we will use $(\mathscr{L}^2)$ to denote the L^2 space of $(\mathscr{S}'(\mathbb{R}), \mathscr{B}(\mathscr{S}'(\mathbb{R})), \mu)$. Note the fact that $\mu(\mathscr{S}_{-p}(\mathbb{R}^d)) = 1$ for any $p > \frac{1}{2}$.

Each $\varphi \in (\mathscr{L}^2)$ possesses a so-called Wiener–Ito's chaos decomposition (cf. Hida, 1980):

$$\varphi(x) = \sum_{n=0}^{\infty} \langle : x^{\otimes n} :, \varphi^{(n)} \rangle$$

for a sequence of $\varphi^{(n)} \in \widehat{L^2(\mathbb{R}^n)}$ ($\widehat{\ }$ stands for symmetrization). In this case,

$$\|\varphi\|_2^2 = \sum_{n=0}^{\infty} n! |\varphi^{(n)}|_2^2,$$

where $\|\varphi\|_2$ denotes the $(\mathscr{L}^2)$-norm of φ. Let $(\mathscr{S})^* \supset (\mathscr{L}^2) \supset (\mathscr{S})$ be the Gelfand's triple of white noise functionals, i.e. $(\mathscr{S})$ is the space of Hida test functionals, and $(\mathscr{S})^*$ is the space of generalized white noise functionals. An element in $(\mathscr{S})^*$ is also called a Hida functional. Let $\Gamma(A)$ be the second quantization of A. For each $p \geqslant 0$, define

$$\|\varphi\|_{2,p} \triangleq \|\Gamma(A)^p \varphi\|_{2,p} \quad \forall \varphi \in (\mathscr{L}^2), \tag{1.1}$$

and let

$$(\mathscr{S})_p \triangleq \{\varphi \in (\mathscr{L}^2)\colon \|\varphi\|_{2,p} < \infty\}, \qquad (\mathscr{S}) = \bigcap_{p \geqslant 0} (\mathscr{S})_p. \tag{1.2}$$

Denote by $(\mathscr{S})^*$ (resp. $(\mathscr{S})_{-p}$) the dual space of $(\mathscr{S})$ (resp. $(\mathscr{S})_p$), and denote by $\langle\langle\cdot,\cdot\rangle\rangle$ the pairing between $(\mathscr{S})_{-p}$ and $(\mathscr{S})_p$ for each $p\in\mathbb{R}$ and also between $(\mathscr{S})^*$ and $(\mathscr{S})$ as well. For each $\varphi\in(\mathscr{S})^*$, its S-transform $S\varphi$ is a (non-linear) functional on $\mathscr{S}(\mathbb{R})$ defined by

$$(S\varphi)(\xi) = \langle\langle\varphi, \mathscr{E}(\xi)\rangle\rangle, \quad \xi\in\mathscr{S}(\mathbb{R}), \tag{1.3}$$

where

$$\mathscr{E}(\xi)(x) =: \mathrm{e}^{\langle x,\xi\rangle} : \hat{=} \exp\left\{\langle x,\xi\rangle - \tfrac{1}{2}|\xi|_2^2\right\} \quad \forall x\in\mathscr{S}'(\mathbb{R}).$$

In particular if $\varphi\in(\mathscr{L}^2)$, then

$$\begin{aligned}(S\varphi)(\xi) &= \int_{\mathscr{S}'(\mathbb{R})} \varphi(x+\xi)\,\mathrm{d}\mu(x)\\ &= \int_{\mathscr{S}'(\mathbb{R})} \varphi(x)\exp\left\{\langle x,\xi\rangle - \tfrac{1}{2}|\xi|_2^2\right\}\,\mathrm{d}\mu(x).\end{aligned}$$

Each $\varphi\in(\mathscr{S})_p$ has a unique Wiener–Ito's decomposition:

$$\varphi(x) = \sum_{n=0}^{\infty}\langle : x^{\otimes n} :, \varphi^{(n)}\rangle \quad \text{for some } \varphi^{(n)}\in\mathscr{S}_p(\hat{\ }\mathbb{R}^n) \tag{1.4}$$

and

$$(S\varphi)(\xi) = \sum_{n=0}^{\infty}\langle\varphi^{(n)}, \xi^{\otimes n}\rangle \quad \forall\xi\in(\mathscr{S}). \tag{1.5}$$

The S-transform $U = S\varphi$ of a Hida functional φ is a so-called U-functional (see Potthoff and Streit, 1991) i.e. the map $\lambda\to U(\xi+\lambda\eta)$ is analytic for any $\xi,\eta\in\mathscr{S}(\mathbb{R})$, and the module $|U(\xi)|\leqslant C_1\exp(C_2|\xi|_p^2)$ for some $p\in\mathbb{R}$ and two nonnegative constants C_1, C_2. Conversely, any U-functional is the S-transform of a unique Hida functional, cf. Hida et al. (1993) and Potthoff and Streit (1991).

According to a theorem of Yokoi (1990), each $F\in(\mathscr{S})$ possesses a unique continuous version, and therefore we will assume that every $F\in(\mathscr{S})$ is continuous.

A generalized white noise functional $\varphi\in(\mathscr{S})^*$ is called a positive functional and denoted by $\varphi\in(\mathscr{S})^*_+$, if $\langle\langle\varphi,F\rangle\rangle\geqslant 0$ for any nonnegative Hida test functional $F\in(\mathscr{S})$. By Yokoi (1990), for each $\varphi\in(\mathscr{S})^*_+$, there is a unique finite measure μ_φ on $(\mathscr{S}'(\mathbb{R}), \mathscr{B}(\mathscr{S}'(\mathbb{R})))$ such that

$$\mu_\varphi(F) = \langle\langle\varphi, F\rangle\rangle \quad \forall F\in(\mathscr{S}).$$

In this case, φ is called the generalized Radon–Nikodym's derivative of the measure μ_φ with respect to the white noise measure μ, and denoted by $\mathrm{d}\mu_\varphi/\mathrm{d}\mu = \varphi$. In particular, each nonnegative, square integrable function belongs to $(\mathscr{S})^*_+$. For each $\lambda\in\mathbb{R}$, $\lambda\neq 0$, define a finite measure $\mu_\varphi^{(\lambda)}$ on $(\mathscr{S}'(\mathbb{R}),\mathscr{B}(\mathscr{S}'(\mathbb{R})))$ by

$$\mu_\varphi^{(\lambda)}(B) \hat{=} \mu_\varphi\left(\frac{B}{\lambda}\right) \quad \forall B\in\mathscr{B}(\mathscr{S}'(\mathbb{R})).$$

We need Lemmas 1.1–1.3; for their proofs, see Potthoff and Yan (1993).

Lemma 1.1. *Let* $\varphi \in (\mathscr{S})_+^*$, $\lambda \in \mathbb{R}$, $\lambda \neq 0$. *Then* $\mathrm{d}\mu_\varphi^{(\lambda)}/\mathrm{d}\mu \in (\mathscr{S})_+^*$ *and*

$$\left(S\frac{\mathrm{d}\mu_\varphi^{(\lambda)}}{\mathrm{d}\mu}\right)(\xi) = e^{-(1/2)(1-\lambda^2)|\xi|_2^2}(S\varphi)(\lambda\xi) \quad \forall \xi \in \mathscr{S}(\mathbb{R}^d). \tag{1.6}$$

For $y \in \mathscr{S}'(\mathbb{R})$, define a measure $\mu_\varphi^{(y)}$ by

$$\mu_\varphi^{(y)}(B) = \mu_\varphi(B - y) \quad \forall B \in \mathscr{B}(\mathscr{S}'(\mathbb{R})).$$

Lemma 1.2. *Let* $y \in \mathscr{S}'(\mathbb{R})$ *and let* $\varphi \in (\mathscr{S})_+^*$. *Then* $\mathrm{d}\mu_\varphi^{(y)}/\mu \in (\mathscr{S})_+^*$ *and*

$$\left(S\frac{\mathrm{d}\mu_\varphi^{(y)}}{\mathrm{d}\mu}\right)(\xi) = \mathrm{e}^{\langle y,\xi\rangle}(S\varphi)(\xi) \quad \forall \xi \in \mathscr{S}(\mathbb{R}). \tag{1.7}$$

In general, for $y \in \mathscr{S}'(\mathbb{R})$, $\lambda \in \mathbb{R}$, $\lambda \neq 0$, define two finite measures $\mu_\varphi^{(\lambda,y)}$ and $\mu_\varphi^{\langle y,\lambda\rangle}$ by

$$\mu_\varphi^{(\lambda,y)}(B) = \mu_\varphi\left(\frac{B}{\lambda} - y\right) \quad \text{and} \quad \mu_\varphi^{\langle y,\lambda\rangle}(B) = \mu_\varphi\left(\frac{B-y}{\lambda}\right),$$

respectively. Then we have

$$\left(S\frac{\mathrm{d}\mu_\varphi^{(\lambda,y)}}{\mathrm{d}\mu}\right)(\xi) = \mathrm{e}^{\lambda\langle y,\xi\rangle}\mathrm{e}^{-(1/2)(1-\lambda^2)|\xi|_2^2}(S\varphi)(\lambda\xi) \tag{1.8}$$

and

$$\left(S\frac{\mathrm{d}\mu_\varphi^{\langle y,\lambda\rangle}}{\mathrm{d}\mu}\right)(\xi) = \mathrm{e}^{\langle y,\xi\rangle}\mathrm{e}^{-(1/2)(1-\lambda^2)|\xi|_2^2}(S\varphi)(\lambda\xi).$$

Lemma 1.3. *Let* ν_1 *and* ν_2 *be two finite measures on* $(\mathscr{S}'(\mathbb{R}), \mathscr{B}(\mathscr{S}'(\mathbb{R})))$ *such that* $\mathrm{d}\nu_1/\mathrm{d}\mu$, $\mathrm{d}\nu_2/\mathrm{d}\mu \in (\mathscr{S})_+^*$. *Then* $\mathrm{d}(\nu_1 * \nu_2)/\mathrm{d}\mu \in (\mathscr{S})_+^*$ *and*

$$\left(S\frac{\mathrm{d}(\nu_1 * \nu_2)}{\mathrm{d}\mu}\right)(\xi) = \mathrm{e}^{(1/2)|\xi|_2^2}\left(S\frac{\mathrm{d}\nu_1}{\mathrm{d}\mu}\right)(\xi)\left(S\frac{\mathrm{d}\nu_2}{\mathrm{d}\mu}\right)(\xi) \quad \forall \xi \in \mathscr{S}(\mathbb{R}). \tag{1.9}$$

2. An integral representation formula

We begin with the definition of the Ornstein–Uhlenbeck semigroup $(P_t)_{t\geqslant 0}$ on the white noise space $(\mathscr{S}'(\mathbb{R}), \mathscr{B}(\mathscr{S}'(\mathbb{R})), \mu)$. By definition,

$$P_t\varphi(x) = \int_{\mathscr{S}'(\mathbb{R})} \varphi(\mathrm{e}^{-t}x + \sqrt{1-\mathrm{e}^{-2t}}\,y)\mu(\mathrm{d}y) \quad \forall \varphi \in (\mathscr{S}). \tag{2.1}$$

$(P_t)_{t\geqslant 0}$ can be uniquely extended to be a μ-symmetric, contraction, strongly continuous, linear operator semigroup on $(\mathscr{L}^2)$ (cf. Yan, 1991; Ikeda and Watanabe, 1989).

Lemma 2.1. *Let $\varphi \in (\mathscr{S})_+^*$ and let μ_φ be the associated finite measure, i.e. $\mu_\varphi(F) = \langle\langle \varphi, F \rangle\rangle$ for any $F \in (\mathscr{S})$. Then for each $t \geqslant 0$, $\mathrm{d}(\mu_\varphi P_t)/\mathrm{d}\mu \in (\mathscr{S})_+^*$ and*

$$\left(S\frac{\mathrm{d}(\mu_\varphi P_t)}{\mathrm{d}\mu}\right)(\xi) = (S\varphi)(\mathrm{e}^{-t}\xi) \quad \forall \xi \in \mathscr{S}(\mathbb{R}). \tag{2.2}$$

Proof. For $F \in (\mathscr{S})$ and $F \geqslant 0$ we have

$$\begin{aligned}(\mu_\varphi P_t)(F) &= \int_{\mathscr{S}'(\mathbb{R})}\int_{\mathscr{S}'(\mathbb{R})} F(\mathrm{e}^{-t}x + y)\mu^{(\sqrt{1-\mathrm{e}^{-2t}})}(\mathrm{d}y)\mu_\varphi(\mathrm{d}x) \\ &= \int_{\mathscr{S}'(\mathbb{R})\times\mathscr{S}'(\mathbb{R})} F(x+y)\mu^{(\sqrt{1-\mathrm{e}^{-2t}})}(\mathrm{d}y)\mu_\varphi^{(\mathrm{e}^{-t})}(\mathrm{d}x),\end{aligned}$$

i.e.,

$$\mu_\varphi P_t = \mu^{(\sqrt{1-\mathrm{e}^{-2t}})} * \mu_\varphi^{(\mathrm{e}^{-t})}. \tag{2.3}$$

Hence, $\mathrm{d}(\mu_\varphi P_t)/\mathrm{d}\mu \in (\mathscr{S})_+^*$ and

$$\begin{aligned}\left(S\frac{\mathrm{d}(\mu_\varphi P_t)}{\mathrm{d}\mu}\right)(\xi) &= \left(S\frac{\mathrm{d}\mu^{(\sqrt{1-\mathrm{e}^{-2t}})}}{\mathrm{d}\mu}\right)(\xi)\left(S\frac{\mathrm{d}\mu_\varphi^{(\mathrm{e}^{-t})}}{\mathrm{d}\mu}\right)(\xi)\mathrm{e}^{(1/2)|\xi|_2^2} \\ &= (\mathscr{S}\varphi)(\mathrm{e}^{-t}\xi). \quad \square\end{aligned}$$

Let $s \to y_s \in \mathscr{S}'(\mathbb{R})$ be a path in $\mathscr{S}'(\mathbb{R})$, and let $v_s \hat{=} \mu(\cdot - y_s) = \mu^{(y_s)}$. Then we have

$$\left(S\frac{\mathrm{d}(v_s P_t)}{\mathrm{d}\mu}\right)(\xi) = \exp\{\langle y_s, \mathrm{e}^{-t}\xi\rangle\} = \exp\{\langle \mathrm{e}^{-t}y_s, \xi\rangle\},$$

and therefore the family of the measures $(v_s)_{s\in R}$ satisfies the following equation:

$$v_s P_t = v_{s+t} \quad \forall t \geqslant 0,\ s \in \mathbb{R},$$

if and only if $\mathrm{e}^{-t}y_s = y_{s+t}$, i.e. if and only if $y_t = \mathrm{e}^{-t}y_0$. Hence, we have obtained the following:

Lemma 2.2. *Let $s \to y_s \in \mathscr{S}'(\mathbb{R})$, $s \in \mathbb{R}$, and let $v_s = \mu^{(y_s)}$. Then (v_s) is an extrance role, i.e.*

$$v_s P_t = v_{t+s} \quad \forall t \geqslant 0,\ s \in \mathbb{R},$$

if and only if there exists a $y \in \mathscr{S}'(\mathbb{R})$ such that $y_s = \mathrm{e}^{-s}y$.

Definition 2.1. A family $u = (u(t,\cdot))_{t\in R}$ of generalized positive functionals is called a generalized positive parabolic function if

$$v_s P_t = v_{s+t} \quad \forall t \geqslant 0,\ s \in \mathbb{R}, \tag{2.4}$$

where v_t denotes the finite measure associated with the positive functional $u(t,\cdot)$.

Thus for each $z \in \mathscr{S}'(\mathbb{R})$, $u^z \hat{=} (u^z(t,\cdot))_{t\in\mathbb{R}}$ is a positive generalized parabolic function, where

$$u^z(t,\cdot) \hat{=} \frac{\mathrm{d}\mu(\cdot - \mathrm{e}^{-t}z)}{\mathrm{d}\mu} \in (\mathscr{S})^*_+, \quad t\in\mathbb{R}, \tag{2.5}$$

$u^z(t,\cdot)$ is also called a generalized Cameron–Martin's density. It is clear that the S-transform of $u^z(t,\cdot)$ is given by

$$(Su^z(t,\cdot))(\xi) = \exp\{\langle \mathrm{e}^{-t}z, \xi\rangle\} \quad \forall t\in\mathbb{R},\ \xi\in\mathscr{S}(\mathbb{R}). \tag{2.6}$$

In particular, if $h\in L^2(\mathbb{R})$, then $\mathrm{d}\mu(\cdot - \mathrm{e}^{-t}h) \ll \mathrm{d}\mu$ and

$$u^h(t,\cdot) = \frac{\mathrm{d}\mu(\cdot - \mathrm{e}^{-t}h)}{\mathrm{d}\mu} = \exp\left\{\langle \cdot, \mathrm{e}^{-t}h\rangle - \frac{\mathrm{e}^{-2t}}{2}|h|_2^2\right\}. \tag{2.7}$$

The family $\{u^h\colon h \in L^2(\mathbb{R})\}$ was used by Röckner (1992) for proving his integral representation theorem for positive parabolic function under some mild conditions.

Theorem 2.1. *Let $u = (u(t,\cdot))_{t\in R}$ be a generalized parabolic function, and let $(v_t)_{t\in\mathbb{R}}$ be the finite measure family on $(\mathscr{S}'(\mathbb{R}), \mathscr{B}(\mathscr{S}'(\mathbb{R})))$ such that $\mathrm{d}v_t/\mathrm{d}\mu = u(t,\cdot)$ for each $t\in\mathbb{R}$. Then we have the following conclusions*:

1. *The functional $\xi \to (Su(t,\cdot))(\mathrm{e}^t\xi)$, $\xi\in\mathscr{S}(\mathbb{R})$, is independent of t.*
2. *Let*

$$G(\xi) = (Su(0,\cdot))(\xi)\mathrm{e}^{-(1/2)|\xi|_2^2}. \tag{2.8}$$

Then there is a unique generalized positive functional $\psi\in(\mathscr{S})^*_+$ such that $S\psi = G$.

3. *Let m_u denote the associated finite measure of the positive generalized white noise functional ψ, i.e. $m_u(F) = \langle\langle \psi, F\rangle\rangle$ for any $F\in(\mathscr{S})$. Then*

$$v_t = \int_{\mathscr{S}'(\mathbb{R})} \mu(\cdot - \mathrm{e}^{-t}z) m_u(\mathrm{d}z) \quad \forall t\in\mathbb{R}, \tag{2.9}$$

i.e. for any positive Hida test functional $F\in(\mathscr{S})$, $z \to \mu^{(\mathrm{e}^{-t}z)}(F)$ is $\mathscr{B}(\mathscr{S}'(\mathbb{R}))$-measurable and

$$v_t(F) = \int_{\mathscr{S}'(\mathbb{R})} \mu^{(\mathrm{e}^{-t}z)}(F) m_u(\mathrm{d}z) \quad \forall t\in\mathbb{R}.$$

Remark. For the Ornstein–Uhlenbeck semigroup on an abstract Wiener space and under the assumption that (v_t) is an entrance law, but without the condition that $\mathrm{d}v_t/\mathrm{d}\mu\in(\mathscr{S})^*_+$, Röckner obtained the above representation theorem. But here we give a precise formula for the representation measure m_u. We also note that, by the formula for m_u, generally the measure m_u is not absolutely continuous with respect to the Gaussian measure μ.

Proof. In fact, for any $\xi\in\mathscr{S}(\mathbb{R})$, we have

$$(Su(t+s,\cdot))(\mathrm{e}^{t+s}\xi) = \left(S\frac{\mathrm{d}(v_tP_s)}{\mathrm{d}\mu}\right)(\mathrm{e}^{s+t}\xi)$$

$$
\begin{aligned}
&= \left(S\frac{\mathrm{d}\mu^{(\sqrt{1-\mathrm{e}^{-2s}})}}{\mathrm{d}\mu} \right)(\mathrm{e}^{s+t}\xi)(Su(t,\cdot)^{(\mathrm{e}^{-t})})(\mathrm{e}^{s+t}\xi)\mathrm{e}^{(1/2)(t+s)^2|\xi|_2^2} \\
&= \mathrm{e}^{-(1/2)\mathrm{e}^{-2s}(t+s)^2|\xi|_2^2}\mathrm{e}^{-(1/2)(1-\mathrm{e}^{-2s})(t+s)^2|\xi|_2^2}(Su(t,\cdot))(\mathrm{e}^t\xi) \\
&\quad \times \mathrm{e}^{(1/2)(t+s)^2|\xi|_2^2} \\
&= (Su(t,\cdot))(\mathrm{e}^t\xi).
\end{aligned}
$$

Thus, we have proved the first conclusion.

It is obvious that G is a U-functional in the sense of Potthoff and Streit (1991) and Hida et al. (1993), and therefore there is a unique $\psi \in (\mathscr{S})^*$ such that its S-transform is G. We now have to show that ψ is a positive functional. The key step is to show that the T-transform (see Hida et al., 1993) of ψ is positively defined.

Given a generalized functional $\varphi \in (\mathscr{S})^*$, the T-transform of φ is defined by

$$T\varphi(\xi) = S\varphi(\mathrm{i}\xi)\mathrm{e}^{-(1/2)|\xi|_2^2} \quad \forall \xi \in \mathscr{S}(\mathbb{R}).$$

Note that

$$Tu(t,\cdot)(\xi) = Su(t,\cdot)(\mathrm{i}\xi)\mathrm{e}^{-(1/2)|\xi|_2^2}$$

is positively defined, and that

$$Su(t,\cdot)(\mathrm{i}\mathrm{e}^t\xi) = Su(0,\cdot)(\mathrm{i}\xi) \quad \forall t \in \mathbb{R}.$$

Therefore,

$$
\begin{aligned}
T\psi(\xi) &= Su(0,\cdot)(\mathrm{i}\xi) = \lim_{t\to-\infty} Su(0,\cdot)(\mathrm{i}\xi)\mathrm{e}^{-(1/2)|\mathrm{e}^t\xi|_2^2} \\
&= \lim_{t\to-\infty} Su(t,\cdot)(\mathrm{i}\mathrm{e}^t\xi)\mathrm{e}^{-(1/2)|\mathrm{e}^t\xi|_2^2}
\end{aligned}
$$

is positively defined, so that $\psi \in (\mathscr{S})^*_+$. We will denote the measure μ_ψ by m_u for an obvious reason.

Now we prove 3. It is clear that $z \to \mu^{(\mathrm{e}^{-t}z)}(F)$ is $\mathscr{B}(\mathscr{S}'(\mathbb{R}))$-measurable for each $F \in (\mathscr{S})$. Let

$$\mu_t \hat{=} \int_{\mathscr{S}'(\mathbb{R})} \mu(\cdot - \mathrm{e}^{-t}z)m_u(\mathrm{d}z) = \int_{\mathscr{S}'(\mathbb{R})} \mu^{(\mathrm{e}^{-t}z)}m_u(\mathrm{d}z).$$

Then for any $\xi \in \mathscr{S}(\mathbb{R})$, we have

$$
\begin{aligned}
\mu_t(\mathscr{E}(\xi)) &= \int_{\mathscr{S}'(\mathbb{R})} \mu^{(\mathrm{e}^{-t}z)}(\mathscr{E}(\xi))m_u(\mathrm{d}z) \\
&= \int_{\mathscr{S}'(\mathbb{R})} \left(S\frac{\mathrm{d}\mu^{(\mathrm{e}^{-t}z)}}{\mathrm{d}\mu} \right)(\xi)m_u(\mathrm{d}z) \\
&= \int_{\mathscr{S}'(\mathbb{R})} \mathrm{e}^{\langle \mathrm{e}^{-t}z,\xi\rangle}m_u(\mathrm{d}z) \\
&= \exp\left\{ \frac{1}{2}\mathrm{e}^{-2t}|\xi|_2^2| \right\} \int_{\mathscr{S}'(\mathbb{R})} \exp\left\{ \langle z, \mathrm{e}^{-t}\xi\rangle - \frac{1}{2}\mathrm{e}^{-2t}|\xi|_2^2 \right\} m_u(\mathrm{d}z)
\end{aligned}
$$

$$= \exp\left\{\frac{1}{2}\mathrm{e}^{-2t}|\xi|_2^2\right\}\left(S\frac{\mathrm{d}m_u}{\mathrm{d}\mu}\right)(\mathrm{e}^{-t}\xi)$$
$$= \exp\{\tfrac{1}{2}\mathrm{e}^{-2t}|\xi|_2^2\}(Su(t,\cdot))(\xi)\exp\{-\tfrac{1}{2}\mathrm{e}^{-2t}|\xi|_2^2\}$$
$$= (Su(t,\cdot))(\xi),$$

i.e. $\mathrm{d}\mu_t/\mathrm{d}\mu \in (\mathscr{S})^*$ and

$$\frac{\mathrm{d}\mu_t}{\mathrm{d}\mu} = u(t,\cdot).$$

Hence $v_t = \mu_t$ for each $t \in \mathbb{R}$, i.e.

$$v_t(\cdot) = \int_{\mathscr{S}'(\mathbb{R})} \mu(\cdot - \mathrm{e}^{-t}z)m_u(\mathrm{d}z). \qquad \square$$

Theorem 2.2. *Let $u = (u(t,\cdot))_{t\in\mathbb{R}}$ be a positive generalized parabolic function on $\mathscr{S}'(\mathbb{R})$. Then*

$$u(t,\cdot) = \int_{\mathscr{S}'(\mathbb{R})} u^z(t,\cdot)m_u(\mathrm{d}z), \tag{2.10}$$

where m_u be the unique finite measure on $(\mathscr{S}'(\mathbb{R}), \mathscr{B}(\mathscr{S}'(\mathbb{R})))$ such that

$$\left(S\frac{\mathrm{d}m_u}{\mathrm{d}\mu}\right)(\xi) = (Su(0,\cdot))(\xi)\mathrm{e}^{-(1/2)|\xi|_2^2}, \quad \xi \in \mathscr{S}'(\mathbb{R}). \tag{2.11}$$

Remark. Röckner (1992) has established a representation theorem for $\mathscr{L}^2$-integrable parabolic function under the assumption that $m_u(L^2(\mathbb{R})) = m_u(\mathscr{S}'(\mathbb{R}))$.

Proof. For any Hida's test functional $F \in (\mathscr{S})$, by Theorem 2.1, we have

$$v_t(F) = \int_{\mathscr{S}'(\mathbb{R})} \mu^{(\mathrm{e}^{-t}z)}(F)m_u(\mathrm{d}z),$$

i.e.

$$\left\langle\!\!\left\langle \frac{\mathrm{d}v_t}{\mathrm{d}\mu}, F \right\rangle\!\!\right\rangle = \int_{\mathscr{S}'(\mathbb{R})} \left\langle\!\!\left\langle \frac{\mathrm{d}\mu^{(\mathrm{e}^{-t}z)}}{\mathrm{d}\mu}, F \right\rangle\!\!\right\rangle m_u(\mathrm{d}z)$$
$$= \left\langle\!\!\left\langle \int_{\mathscr{S}'(\mathbb{R})} \frac{\mathrm{d}\mu^{(\mathrm{e}^{-t}z)}}{\mathrm{d}\mu} m_u(\mathrm{d}z), F \right\rangle\!\!\right\rangle.$$

Hence,

$$u(t,\cdot) = \int_{\mathscr{S}'(\mathbb{R})} u^z(t,\cdot)m_u(\mathrm{d}z). \qquad \square$$

In particular, if $u = (u(t,\cdot))_{t\in\mathbb{R}}$ is a positive $(\mathscr{L}^2)$-integrable parabolic function, i.e. each $u(t,\cdot) \in (\mathscr{L}^2)$ and $v_t = u(t,\cdot), \mu$ satisfying

$$v_t P_s = v_{t+s} \quad \forall t \in \mathbb{R},\ s \geqslant 0,$$

then $u = (u(t,\cdot))_{t\in R}$ is a positive generalized parabolic function and the associated measure of $u(t,\cdot)$ is ν_t. By Theorem 2.2., we have

$$u(t,\cdot) = \int_{\mathscr{S}'(\mathbb{R})} u^z(t,\cdot)m_u(\mathrm{d}z).$$

The assumption suggested by Röckner (1992) implies that $m_u(L^2(\mathbb{R})) = m_u(\mathscr{S}'(\mathbb{R}))$. On the other hand, if $h\in L^2(\mathbb{R})$, then

$$u^h(t,\cdot) = \exp(\langle\cdot,\mathrm{e}^{-t}h\rangle - \tfrac{1}{2}\mathrm{e}^{-2t}|h|_2^2).$$

Hence, we have

$$u(t,\cdot) = \int_{L^2(\mathbb{R})} \exp[\langle\cdot,\mathrm{e}^{-t}h\rangle - \tfrac{1}{2}\mathrm{e}^{-2t}|h|_2^2]m_u(\mathrm{d}h).$$

Thus, we reobtained the representation form established by Röckner (1992).

Example. Let $x\in\mathscr{S}'(\mathbb{R})$ and $u(t,\cdot) = u^x(t,\cdot)$. Then u is a generalized positive parabolic function. By Theorem 2.1, the S-transform of the associated measure m_u is

$$G(\xi) = (Su(0,\cdot))(\xi)\mathrm{e}^{-(1/2)|\xi|_2^2} = \exp\{\langle x,\xi\rangle - \tfrac{1}{2}|\xi|_2^2\},$$

and therefore m_u is the evolution measure at x as we expected.

3. Parabolic functions in $(\mathscr{S})_p$

Let $u=(u(t,\cdot))_{t\in\mathbb{R}}$ be an $(\mathscr{L}^2)$-integrable positive parabolic function. Then

$$u(t,\cdot) = \int_{\mathscr{S}'(\mathbb{R})} u^z(t,\cdot)m_u(\mathrm{d}z),$$

where m_u is a finite measure on $(\mathscr{S}'(\mathbb{R}),\mathscr{B}(\mathscr{S}'(\mathbb{R})))$. Thus,

$$u(t,\cdot) = \int_{L^2(\mathbb{R})} u^h(t,\cdot)m_u(\mathrm{d}h)$$

with $u^h(t,\cdot)=\exp\{\langle\cdot,\mathrm{e}^{-t}h\rangle - \tfrac{1}{2}\mathrm{e}^{-2t}|h|_2^2\}$ if and only if $m_u(L^2(\mathbb{R}))=m_u(\mathscr{S}'(\mathbb{R}))$. In general, even for an $(\mathscr{L}^2)$-integrable positive parabolic function, the condition that $m_u(L^2(\mathbb{R}))=m_u(\mathscr{S}'(\mathbb{R}))$ is not always satisfied. Such examples are given in Röckner (1992).

Lemma 3.1. *Let $\varphi\in(\mathscr{S})_p$, $p\in\mathbb{R}$. Then the functional*

$$F(\xi) = \langle\langle\varphi,\mathscr{E}(\mathrm{i}\xi)\rangle\rangle \quad \forall\xi\in\mathscr{S}(\mathbb{R})$$

is continuous in $|\cdot|_{2,-p}$-norm.

Proof. $\mathscr{E}(\mathrm{i}\xi)$ possesses Wiener–Ito's chaos decomposition

$$\mathscr{E}(\mathrm{i}\xi)(x) = \sum_{n=0}^{\infty}\left\langle :x^{\otimes n}:, \frac{\mathrm{i}^n\xi^{\otimes n}}{n!}\right\rangle, \quad \xi\in\mathscr{S}(\mathbb{R}).$$

Thus,

$$\|\mathscr{E}(\mathrm{i}\xi)-\mathscr{E}(\mathrm{i}\eta)\|_{2,-p}^2=\sum_{n=0}^{\infty}\frac{1}{n!}|\xi^{\otimes n}-\eta^{\otimes n}|_{2,-p}^2 \quad \forall \xi,\eta\in\mathscr{S}(\mathbb{R}).$$

However,

$$|\xi^{\otimes n}-\eta^{\otimes n}|_{2,-p}^2=|\xi|_{2,-p}^{2n}-2\langle\xi,\eta\rangle_{2,-p}^n+|\eta|_{2,-p}^{2n},$$

and therefore

$$\begin{aligned}|F(\xi)-F(\eta)|^2 &\leqslant \|\varphi\|_{2,p}^2\|\mathscr{E}(\mathrm{i}\xi)-\mathscr{E}(\mathrm{i}\eta)\|_{2,-p}^2\\ &=\|\varphi\|_{2,p}^2[\mathrm{e}^{|\xi|_{2,-p}^2}-2\mathrm{e}^{\langle\xi,\eta\rangle_{2,-p}}+\mathrm{e}^{|\eta|_{2,-p}^2}]\\ &\to 0 \quad (\text{as } \eta\to\xi \text{ in } |\cdot|_{2,-p}\text{-norm}). \qquad \square\end{aligned}$$

Proposition 3.1. *Let* $u=(u(t,\cdot))_{t\in\mathbb{R}}$ *be an* $(\mathscr{L}^2)$*-integrable positive parabolic function. Then for any* $p>\frac{1}{2}$,

$$u(t,\cdot)=\int_{\mathscr{S}_{-p}(\mathbb{R})}u^z(t,\cdot)m_u(\mathrm{d}z),$$

where $\mathrm{d}m_u/\mathrm{d}\mu\in(\mathscr{S})_+^*$ *and*

$$S\frac{\mathrm{d}m_u}{\mathrm{d}\mu}(\xi)=Su(0,\cdot)(\xi)\mathrm{e}^{-(1/2)|\xi|_2^2}.$$

Proof. In fact, we only have to prove $m_u(\mathscr{S}'(\mathbb{R}))=m_u(\mathscr{S}_{-p}(\mathbb{R}))$. Note that the characteristic functional $C_u(\xi)$ of the finite measure m_u is given by

$$\begin{aligned}C_u(\xi)&=Su(0,\cdot)(\mathrm{i}\xi),\\ &=\langle\langle u(0,\cdot),\mathscr{E}(\mathrm{i}\xi)\rangle\rangle \quad \forall\xi\in\mathscr{S}(\mathbb{R}).\end{aligned}$$

By the assumption that $u(0,\cdot)\in(\mathscr{L}^2)$ implies that the functional $\xi\to\langle\langle u(0,\cdot),\mathscr{E}(\mathrm{i}\xi)\rangle\rangle$ is continuous in $L^2(\mathbb{R})$-norm. By Bochner–Minlos' theorem (cf. Hida, 1980), we know that $m_u(\mathscr{S}'(\mathbb{R}))=m_u(\mathscr{S}_{-p}(\mathbb{R}))$. $\square$

Remark. For each $p>\frac{1}{2}$, $(\mathscr{S}_{-p}(\mathbb{R}),L^2(\mathbb{R}),\mu)$ is an abstract Wiener space. Proposition 2.1 shows that any square integrable, positive, parabolic function has an integral representation via the generalized Cameron–Martin density functions indexed by the space $\mathscr{S}_{-p}(\mathbb{R})$. For a general abstract Wiener space (W,H,μ), the above integral representation strongly suggests that

$$u(t,\cdot)=\int_W u^h(t,\cdot)m_u(\mathrm{d}h)$$

for any square integrable positive parabolic function of the Ornstein–Uhlenbeck operator on the abstract Wiener space (W,H,μ). To this end, one should first recognize the "density functions" $u^z(t,\cdot)$ for any $z\in W$.

Theorem 3.1. *Let* $u=(u(t,\cdot))_{t\in\mathbb{R}}$ *be a positive generalized parabolic function with* $u(0,\cdot)\in(\mathscr{S})_p$ *for some* $p>\frac{1}{2}$. *Then*

$$u(t,\cdot)=\int_{\mathscr{S}_{(p-q)}}\exp\left[\langle\cdot,\mathrm{e}^{-t}h\rangle-\frac{\mathrm{e}^{-2t}}{2}|h|_2^2\right]m_u(\mathrm{d}h) \tag{3.1}$$

for any q *such that* $\frac{1}{2}<q<p$.

Proof. We have

$$u(t,\cdot)=\int_{\mathscr{S}'(\mathbb{R})}u^z(t,\cdot)m_u(\mathrm{d}z).$$

The characteristic functional $C_u(\xi)$ of m_u is given

$$C_u(\xi)=\langle\langle u(0,\cdot),\mathscr{E}(\mathrm{i}\xi)\rangle\rangle.$$

By the assumption that $u(0,\cdot)\in(\mathscr{S})_p$, $\xi\to C_u(\xi)$ is continuous in the norm $|\cdot|_{2,-p}$. For $\frac{1}{2}<q<p$, the injection $\mathscr{S}_{-p+q}(\mathbb{R})\to\mathscr{S}_{-p}(\mathbb{R})$ is of Hilbert–Schmidt type. By Bochner–Minlos's theorem (cf. Hida, 1980, p. 121, Theorem 3.1), we have

$$m_u(\mathscr{S}_{-(-p+q)}(\mathbb{R}))=m_u(\mathscr{S}_{(p-q)}(\mathbb{R}))=m_u(\mathscr{S}'(\mathbb{R})).$$

However $p-q>0$, so that $\mathscr{S}_{p-q}(\mathbb{R})\subset L^2(\mathbb{R})$, and therefore the conclusion follows from the fact that

$$u^h(t,\cdot)=\exp\left[\langle\cdot,\mathrm{e}^{-t}h\rangle-\frac{\mathrm{e}^{-2t}}{2}|h|_2^2\right]\quad\forall h\in L^2(\mathbb{R}).\qquad\square$$

By the proof of Theorem 2.1, we know that if $u=(u,t,\cdot))_{t\in\mathbb{R}}$ is a positive generalized parabolic function with $u(0,\cdot)\in(\mathscr{S})_p$ for some $p\in\mathbb{R}$, then for any $q>\frac{1}{2}$,

$$u(t,\cdot)=\int_{\mathscr{S}_{(p-q)}(\mathbb{R})}u^z(t,\cdot)m_u(\mathrm{d}z).$$

Corollary 3.1. *Let* $u=(u(t,\cdot))_{t\in\mathbb{R}}$ *be a positive generalized parabolic function with* $u(0,\cdot)\in(\mathscr{S})_p$ *for some* $p>1$. *For any* q *such that* $\frac{1}{2}<p-q$, $q>\frac{1}{2}$, *let*

$$\hat{u}(t,x)=\int_{\mathscr{S}_{(p-q)}(\mathbb{R})}\exp\left[\langle x,\mathrm{e}^{-t}h\rangle-\frac{\mathrm{e}^{-2t}}{2}|h|_2^2\right]m_u(\mathrm{d}h),\quad x\in\mathscr{S}_{(q-p)}(\mathbb{R}). \tag{3.2}$$

Then

$$u(t,\cdot)=\hat{u}(t,\cdot)\quad in\ (\mathscr{L}^2).$$

Proof. We only have to note that if $p-q>\frac{1}{2}$, we have $\mu(\mathscr{S}_{(q-p)}(\mathbb{R}))=1$. $\square$

Corollary 3.2. *Let* $u=(u(t,\cdot))_{t\in\mathbb{R}}$ *be a positive generalized parabolic function with* $u(0,\cdot)\in(\mathscr{S})$. *Then* $m_u(\mathscr{S}(\mathbb{R}))=m_u(\mathscr{S}'(\mathbb{R}))$, *and*

$$\hat{u}(t,x)=\int_{\mathscr{S}(\mathbb{R})}\exp\left\{\langle x,\mathrm{e}^{-t}h\rangle-\frac{\mathrm{e}^{-2t}}{2}|h|_2^2\right\}m_u(\mathrm{d}h),\quad x\in\mathscr{S}'(\mathbb{R}), \tag{3.3}$$

is continuous version of $u(t,\cdot)$.

Remark. For any $\varphi \in (\mathscr{S})^*_+$, we define a positive generalized parabolic function $u = (u,(t,\cdot))_{t\in\mathbb{R}}$ by

$$u(t,\cdot) = \int_{\mathscr{S}'(\mathbb{R})} u^z(t,\cdot)\mu_\varphi(\mathrm{d}z).$$

Indeed, for any $\xi \in \mathscr{S}(\mathbb{R})$,

$$\int_{\mathscr{S}'(\mathbb{R})} \langle\langle u^z(t,\cdot), \mathscr{E}(\xi)\rangle\rangle \mu_\varphi(\mathrm{d}z) = \int_{\mathscr{S}'(\mathbb{R})} \mathrm{e}^{\langle \mathrm{e}^{-t}z,\xi\rangle - (1/2)|\xi|_2^2}\mu_\varphi(\mathrm{d}z)$$

$$= (\mathscr{S}\varphi)(\mathrm{e}^{-t}\xi)\mathrm{e}^{(1/2)(\mathrm{e}^{-2t}-1)|\xi|_2^2},$$

so that $u(t,\cdot) \in (\mathscr{S})^*_+$ (cf. Potthoff and Streit, 1991). It is easily seen that $u = (u(t,\cdot))_{t\in\mathbb{R}}$ is a positive generalized parabolic function.

4. Heat equation

We recall the definition of the Ornstein–Uhlenbeck operator $\mathscr{N}$ on $(\mathscr{S})^*$. Let $\varphi \in (\mathscr{S})^*$ and let

$$\varphi(x) = \sum_{n=0}^{\infty} \langle : x^{\otimes n} :, f_n\rangle \quad \text{for some } f_n \in \mathscr{S}'(\widehat{\mathbb{R}}^n).$$

Define $\mathscr{N}\varphi \in (\mathscr{S})^*$ by

$$(\mathscr{N}\varphi)(x) = \sum_{n=0}^{\infty} n\langle : x^{\otimes n} :, f_n\rangle, \tag{4.1}$$

and call $\mathscr{N}$ the number operator; $\mathscr{N}$ is a continuous operator on $(\mathscr{S})^*$. In fact (see Kuo, 1992),

$$\|\mathscr{N}\varphi\|_{2,-q} \leqslant \|\varphi\|_{2,-p} \quad \forall \varphi \in (\mathscr{S})^*,\ q \geqslant p + \tfrac{2}{3}. \tag{4.2}$$

In particular, for each $z \in \mathscr{S}'(\mathbb{R})$ we have

$$u^z(t,x) = \sum_{n=0}^{\infty} \left\langle : x^{\otimes n} :, \frac{\mathrm{e}^{-nt}z^{\otimes n}}{n!}\right\rangle. \tag{4.3}$$

Hence,

$$\mathscr{N}u^z(t,x) = \sum_{n=1}^{\infty} \left\langle : x^{\otimes n} :, \frac{n\mathrm{e}^{-nt}z^{\otimes n}}{n!}\right\rangle. \tag{4.4}$$

Proposition 4.1. *For $z \in \mathscr{S}_{-p}(\mathbb{R})$, $p \in \mathbb{R}$, $\mathscr{N}u^z(t,\cdot) \in (\mathscr{S})_{-p}$ for each $t \in \mathbb{R}$ and*

$$\frac{\mathrm{d}u^z(t,\cdot)}{\mathrm{d}t} = -\mathscr{N}u^z(t,\cdot) \quad \textit{in } (\mathscr{S})_{-p}. \tag{4.5}$$

Proof. It is easily seen that $\mathcal{N}u^z(t,\cdot)\in(\mathcal{S})_{-p}$. For any $q\geqslant p$, $t\in\mathbb{R}$ and $0<|\Delta t|<1$, we have

$$\begin{aligned}
&\left\|\frac{u^z(t+\Delta t,\cdot)-u^z(t,\cdot)}{\Delta t}-(-\mathcal{N})u^z(t,\cdot)\right\|_{2,-q}^2\\
&=\sum_{n=1}^{\infty} n!\left|\frac{e^{-n(t+\Delta t)}-e^{-nt}}{\Delta t}+ne^{-nt}\right|^2\frac{|z|_{2,-q}^{2n}}{n!^2}\\
&\leqslant\sum_{n=1}^{\infty}\frac{n^2}{n!}e^{n(|t|+1)}(|z|_{2,-q}^2)^n|\Delta t|^2\\
&=\{e^{2(|t|+1)}|z|_{2,-q}^4\exp[e^{(|t|+1)}|z|_{2,-q}^2]+e^{(|t|+1)}|z|_{2,-q}^2\exp[e^{(|t|+1)}|z|_{2,-q}^2]\}|\Delta t|^2.
\end{aligned}$$

In particular, for any $q\geqslant p$,

$$\left\|\frac{u^z(t+\Delta t,\cdot)-u^z(t,\cdot)}{\Delta t}-(-\mathcal{N})u^z(t,\cdot)\right\|_{2,-q}\to 0\quad\text{as }\Delta t\to 0,$$

i.e.,

$$\frac{du^z(t,\cdot)}{dt}=-\mathcal{N}u^z(t,\cdot)\quad\text{in }(\mathcal{S})_{-p}.\qquad\square$$

Theorem 4.1. *Let $u=(u(t,\cdot))_{t\in\mathbb{R}}$ be a positive generalized parabolic function. Then*

$$\frac{du(t,\cdot)}{dt}=-\mathcal{N}u(t,\cdot)\quad\text{in }(\mathcal{S})^*.\tag{4.6}$$

Moreover, there exists a $q(t)\in\mathbb{R}$ for each $t\in\mathbb{R}$, such that

$$\frac{du(t,\cdot)}{dt}=-\mathcal{N}u(t,\cdot)\quad\text{in }(\mathcal{S})_{-q(t)}.$$

Proof. First we note that

$$\mathcal{N}u(t,\cdot)=\int_{\mathcal{S}'(\mathbb{R})}\mathcal{N}u^z(t,\cdot)m_u(dz)$$

by the fact that $N:(\mathcal{S})^*\to(\mathcal{S})^*$ is continuous, i.e., for any $F\in(\mathcal{S})$ we have

$$\langle\langle\mathcal{N}u(t,\cdot),F\rangle\rangle=\int_{\mathcal{S}'(\mathbb{R})}\langle\langle\mathcal{N}u^z(t,\cdot),F\rangle\rangle m_u(dz).$$

By (2.5) we know that there is a $p\in\mathbb{R}$ such that

$$u(t,\cdot)=\int_{\mathcal{S}_{-p}(\mathbb{R})}u^z(t,\cdot)m_u(dz),$$

where m_u is a finite measure on $(\mathcal{S}'(\mathbb{R}),\mathcal{B}(\mathcal{S}'(\mathbb{R})))$ with support $\mathcal{S}_{-p}(\mathbb{R})$. For any $q>p+\frac{2}{3}$, $t\in\mathbb{R}$ and $0<|\Delta t|<1$ we have

$$\left\|\frac{u(t+\Delta t,\cdot)-u(t,\cdot)}{\Delta t}-(-\mathcal{N})u(t,\cdot)\right\|_{2,-q}^2$$

$$\leqslant \int_{\mathscr{S}_{-q}(\mathbb{R})} \left\| \frac{u^z(t+\Delta t,\cdot)-u^z(t,\cdot)}{\Delta t} - (-\mathscr{N})u^z(t,\cdot) \right\|_{2,-q}^2 m_u(\mathrm{d}z)(m_u(\mathscr{S}'(\mathbb{R})))^2$$

$$\leqslant |\Delta t|^2 m_u(\mathscr{S}'(\mathbb{R}))^2 \int_{\mathscr{S}_{-p}(\mathbb{R})} \{\mathrm{e}^{2(|t|+1)}|z|_{2,-q}^4 \exp[\mathrm{e}^{(|t|+1)}|z|_{2,-q}^2]$$

$$+\mathrm{e}^{(|t|+1)}|z|_{2,-q}^2 \exp[\mathrm{e}^{(|t|+1)}|z|_{2,-q}^2]\} m_u(\mathrm{d}z).$$

For each $t \in \mathbb{R}$, there exists a number $q(t)$, such that the integral in the above inequality is finite for $q = q(t)$ (cf. Yan, 1993, Theorem 9.2.). Hence,

$$\frac{\mathrm{d}u(t,\cdot)}{\mathrm{d}t} = -\mathscr{N}u(t,\cdot) \quad \text{in } (\mathscr{S})_{-q(t)}. \qquad \square$$

Proposition 4.2. *Let $u = (u(t,\cdot))_{t\in\mathbb{R}}$ be a positive generalized parabolic function with $u(t,\cdot)\in(\mathscr{S})$. Then*

$$\frac{\mathrm{d}u(t,x)}{\mathrm{d}t} = -\mathscr{N}u(t,x) \quad \forall (t,x)\in\mathbb{R}\times\mathscr{S}'(\mathbb{R}). \tag{4.7}$$

Proof. For any $(t,x)\in\mathbb{R}\times\mathscr{S}'(\mathbb{R})$, we have

$$u(t,x) = \int_{\mathscr{S}(\mathbb{R})} \exp\left\{\langle x, \mathrm{e}^{-t}h\rangle - \frac{\mathrm{e}^{-2t}}{2}|h|_2^2\right\} m_u(\mathrm{d}h).$$

Note that $\mathscr{N}u^h(t,\cdot)\in(\mathscr{S})$ for each $h\in(\mathscr{S})$ and

$$-\mathscr{N}u^h(t,x) = [-\mathrm{e}^{-t}\langle x,h\rangle + \mathrm{e}^{-2t}|h|_2^2]u^h(t,x)$$

$$= \frac{\mathrm{d}u^h(t,x)}{\mathrm{d}t}.$$

Hence, we have

$$\frac{\mathrm{d}u(t,x)}{\mathrm{d}t} = \int_{\mathscr{S}(\mathbb{R})} -\mathscr{N}u^h(t,x)m_u(\mathrm{d}h)$$

$$= -\mathscr{N}\int_{\mathscr{S}(\mathbb{R})} u^h(t,x)m_u(\mathrm{d}h). \qquad \square$$

5. Extreme points of parabolic functions

Denote by $\mathscr{M} = \{\mu_\varphi : \varphi\in(\mathscr{S})_+^*\}$. By Theorem 9.2 in Yan (1993), a finite measure v on $(\mathscr{S}'(\mathbb{R}), \mathscr{B}(\mathscr{S}'(\mathbb{R})))$ belongs to $\mathscr{M}$ if and only if there exist $\alpha > 0$ and $p > \frac{1}{2}$ such that

$$\int_{\mathscr{S}'(\mathbb{R})} \mathrm{e}^{\alpha|x|_{2,-p}^2} v(\mathrm{d}x) < \infty. \tag{5.1}$$

Let $\mathscr{M}^1$ be the set of all the probability measures in $\mathscr{M}$. Then $\mathscr{M}^1$ is a convex set.

Proposition 5.1. *$\nu \in \mathscr{M}^1$ is an extreme point of $\mathscr{M}^1$ if and only if for any $B \in \mathscr{B}(\mathscr{S}'(\mathbb{R}))$,*

$$\nu(B) = 1 \quad \text{or } 0.$$

In particular, for each $z \in \mathscr{S}'(\mathbb{R})$, δ_z is an extreme point of $\mathscr{M}^1$.

Proof. Let $\nu \in \mathscr{M}^1$, such that $\nu(B) = 1$ or 0 for any $B \in \mathscr{B}(\mathscr{S}'(\mathbb{R}))$, and there exist $0 < a < 1$, $\nu_1, \nu_2 \in \mathscr{M}^1$ such that

$$\nu = a\nu_1 + (1-a)\nu_2.$$

Then $\nu(B) = a\nu_1(B) + (1-a)\nu_2(B)$. If $\nu(B) = 1$, then

$$1 = a\nu_1(B) + (1-a)\nu_2(B),$$
$$0 = a\nu_1(B^c) + (1-a)\nu_2(B^c),$$

so that, $\nu_1(B^c) = \nu_2(B^c) = 0$. Hence $\nu = \nu_1 = \nu_2$, and therefore ν is an extreme point. Now assume that $\nu \in \mathscr{M}^1$, and there is a set $B \in \mathscr{B}(\mathscr{S}'(\mathbb{R}))$ such that $0 < \nu(B) < 1$. Let

$$\nu_1(\cdot) = \frac{1}{\nu(B)}\nu(B \cap \cdot), \qquad \nu_2(\cdot) = \frac{1}{\nu(B^c)}\nu(B^c \cap \cdot).$$

Then $\nu_1(\mathscr{S}'(\mathbb{R})) = \nu_2(\mathscr{S}'(\mathbb{R})) = 1$. Let $\alpha > 0$ and $p > \frac{1}{2}$ such that

$$\int_{\mathscr{S}'(\mathbb{R})} e^{\alpha |x|_{2,-p}^2} \nu(dx) < \infty.$$

Then

$$\int_{\mathscr{S}'(\mathbb{R})} e^{\alpha |x|_{2,-p}^2} \nu_1(dx) \leqslant \frac{1}{\nu(B)} \int_{\mathscr{S}'(\mathbb{R})} e^{\alpha |x|_{2,-p}^2} \nu(dx) < \infty.$$

Thus $\nu_1 \in \mathscr{M}^1$. By the same reason, $\nu_2 \in \mathscr{M}^1$. It is obvious that

$$\nu = \nu(B)\nu_1 + \nu(B^c)\nu_2,$$

i.e., ν is not an extreme point. □

Let

$$\mathscr{P}_1 = \{u = u(t,\cdot))_{t\in\mathbb{R}}: u \text{ is a positive generalized parabolic function such that } \langle\langle u(t,\cdot), 1\rangle\rangle = 1\}.$$

Proposition 5.2. *Let $u = (u(t,\cdot))_{t\in\mathbb{R}}$ be a positive generalized parabolic function. Then*

1. *$u \in \mathscr{P}_1$ if and only if $m_u \in \mathscr{M}^1$. In particular, $\langle\langle u(t,\cdot), 1\rangle\rangle$ is independent of t.*
2. *$u \in \mathscr{P}_1$ is an extreme point if and only if m_u is an extreme point of $\mathscr{M}^1$.*
3. *For each $z \in \mathscr{S}'(\mathbb{R})$, $u^z = (u^z(t,\cdot))_{t\in\mathbb{R}}$ is an extreme point.*

Proof. By the integral representation, we have

$$u(t,\cdot) = \int_{\mathscr{S}'(\mathbb{R})} u^z(t,\cdot) m_u(dz).$$

Hence,

$$\langle\langle u(t,\cdot),1\rangle\rangle = \int_{\mathscr{S}'(\mathbb{R})} \langle\langle u^z(t,\cdot),1\rangle\rangle m_u(\mathrm{d}z) = \left\langle\left\langle \frac{\mathrm{d}m_u}{\mathrm{d}\mu},1\right\rangle\right\rangle,$$

i.e., $m_u(\mathscr{S}'(\mathbb{R})) = \langle\langle u(t,\cdot),1\rangle\rangle$, and $u\in\mathscr{P}_1$ if and only if $m_u(\mathscr{S}'(\mathbb{R})) = 1$. Moreover, let $u_1,u_2\in\mathscr{P}_1$. Then

$$u(t,\cdot) = au_1(t,\cdot) + (1-a)u_2(t,\cdot)$$

for each $t\in\mathbb{R}$ if and only if

$$m_u = am_{u_1} + (1-a)m_{u_2}.$$

The conclusions follow immediately. □

Corollary. Let $u = (u(t,\cdot))_{t\in\mathbb{R}}$ be a positive generalized parabolic function such that $\langle\langle u(t,\cdot),1\rangle\rangle = 1$. Then u is an extreme point of $\mathscr{P}^1$ if and only if $u(t,\cdot) = u^z(t,\cdot)$ for some $z\in\mathscr{S}'(\mathbb{R})$.

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